Damping of surface waves in an incompressible liquid

By K. M. CASE and W. C. PARKINSON
Physics Department, University of Michigan, Ann Arbor, Michigan

(*Received* 26 *November* 1956)

SUMMARY

The damping of surface waves of small amplitude in liquid contained in cylinders has been calculated. Viscous dissipation in an assumed laminar boundary layer was taken to be the primary cause of damping. Experimental results were obtained for the logarithmic decrement as a function of the ratio of liquid height to cylinder radius for several water-filled cylinders. Theory and experiment were found to be in good agreement.

I. INTRODUCTION

The calculation of the natural frequencies of the surface waves in a liquid contained within solid boundaries except for one free surface is a well-known and largely solved classical problem (see Lamb (1945), § 257); but the corresponding problem of the damping of these waves does not appear to have been so thoroughly treated. Some early work on the viscous damping of surface waves is reported in Lamb (1945, §§ 348 & 349), but this was mainly confined to progressive waves far removed from solid boundaries. The first attempt to account for the effects of solid boundaries on wave damping appears to have been made by Boussinesq as early as 1878. His theory extends to both progressive and standing waves, and was used in more recent times by Keulegan (1948) as a starting point for a calculation of the attenuation of solitary waves. Other modern work on the damping of progressive waves includes that of Biesel (1949), which deals with waves in a channel of finite depth but infinite width, and that of Ursell (1952), which concerns the dissipation in the vicinity of vertical walls when the depth is infinite. Furthermore, Hunt (1952) has calculated the combined effects of finite width and finite depth. The latter authors all employed boundary-layer approximations, which are applied in this paper to the case of *standing* waves. In particular, we shall give the results of calculations of the damping of standing waves in right circular cylinders.

Since the calculations involve a number of idealizations, it seemed advisable to check the theory with experiment. Accordingly, the damping was measured for water in cylinders of different radii as a function of the ratio of water depth to cylinder radius. The apparatus used and the results obtained are described in § III. In § IV it is shown that, subject to certain limitations, the agreement between the theory and experiments is satisfactory.

II. Analysis

In the interests of simplicity we have restricted ourselves to the consideration of small amplitude oscillations. In addition to linearizing the equations, this restriction frees us from worries about turbulence. Rough calculations indicate that with cylinders of the radii used in the measurements the flow is always far from the turbulent state. Some idea of the limits of validity of the small amplitude approximation can be obtained from the results of the experiments (§ III) by noting what amplitudes of excitation give rise to a simple exponential decay law.

For completeness and to provide a basis for the later work, we first sketch the calculation of the natural frequencies.

Consider a rigid right circular cylinder of radius R with base at $z = -\frac{1}{2}h$. The equilibrium free surface of the liquid in the cylinder is at $z = +\frac{1}{2}h$. For liquids with low viscosity, such as water, we can expect that a good zeroth approximation is obtained by neglecting viscosity entirely and describing the liquid velocity distribution by means of the velocity potential function ϕ_c. Thus, we take

$$\mathbf{u} = -\nabla\phi_c, \tag{1}$$

$$\nabla^2\phi_c = 0, \tag{2}$$

and

$$\frac{\partial\phi_c}{\partial n} = 0 \tag{3}$$

on rigid boundaries, where n is the normal vector.

In cylindrical coordinates (r, θ, z) solutions of Laplace's equation subject to the boundary conditions (3) are

$$\phi_c = [\cosh k_{ms}(z+\tfrac{1}{2}h)]\begin{Bmatrix}\chi_{ms}(r,\theta)\\ \psi_{ms}(r,\theta)\end{Bmatrix}, \tag{4}$$

where

$$\begin{Bmatrix}\chi_{ms}\\ \psi_{ms}\end{Bmatrix} = N_{ms}\begin{Bmatrix}\cos s\theta\\ \sin s\theta\end{Bmatrix}J_s(k_{ms}r). \tag{5}$$

Here,

$$N_{ms} = \left\{\tfrac{1}{2}\pi R^2 J_s^2(k_{ms}R)\left(1-\frac{s^2}{(k_{ms}R)^2}\right)\right\}^{-1/2} \tag{6}$$

is chosen so that the integrals over the cylinder cross-section of χ_{ms}^2 and ψ_{ms}^2 are unity. The k_{ms} are defined by

$$J_s'(k_{ms}R) = 0. \tag{7}$$

Expanding ϕ_c in terms of these proper functions, we have

$$\phi_c = \sum_{ms}\{\cosh k_{ms}(z+\tfrac{1}{2}h\}\{A_{ms}\chi_{ms}+B_{ms}\psi_{ms}\}. \tag{8}$$

If we expand $\eta(r,\theta)$ (the height of the free surface at (r,θ) above the equilibrium plane) in terms of the χ_{ms} and ψ_{ms} we have

$$\eta(r,\theta) = \sum_{ms}\{q_{ms}\chi_{ms}+p_{ms}\psi_{ms}\}. \tag{9}$$

The free surface condition

$$\frac{\partial \eta}{\partial t} = u_z \tag{10}$$

permits us to express the expansion coefficients A_{ms}, B_{ms} in terms of the 'surface coordinates' q_{ms}, p_{ms} in the form

$$\begin{Bmatrix} A_{ms} \\ B_{ms} \end{Bmatrix} = -\begin{Bmatrix} \dot{q}_{ms} \\ \dot{p}_{ms} \end{Bmatrix} \Big/ k_{ms} \sinh k_{ms} h. \tag{11}$$

In these coordinates the kinetic energy becomes

$$T = \int_V \tfrac{1}{2}\rho \mathbf{u}^2 \, dV = \tfrac{1}{2}\rho \sum_{ms} [\dot{q}^2_{ms} + \dot{p}^2_{ms}] \frac{\coth k_{ms} h}{k_{ms}}, \tag{12 a}$$

where V and ρ are the liquid volume and density respectively.

The potential energy Ω of the liquid is the sum of the gravitational potential energy (Ω_g) and the surface tension energy (Ω_σ), where

$$\Omega_g = \rho g \int_V z \, dV = \tfrac{1}{2}\rho g \sum_{ms} [q^2_{ms} + p^2_{ms}], \tag{13 a}$$

and

$$\Omega_\sigma = \int_{S_t} \tfrac{1}{2}\sigma \left[\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 \right] dx dy$$

$$= \tfrac{1}{2}\sigma \sum_{ms} k^2_{ms} [q^2_{ms} + p^2_{ms}]. \tag{14}$$

(Here S_t is the equilibrium free surface $z = \frac{1}{2}h$ and σ is the surface tension coefficient.)

The equations for the normal coordinates q_{ms} or p_{ms} obtained by substituting the energy expressions (12 a), (13 a) and (14) in Lagrange's equations (see Lamb (1945), § 135), are

$$\rho \ddot{q}_{ms}(\coth k_{ms} h)/k_{ms} + (\rho g + \sigma k^2_{ms}) q_{ms} = 0. \tag{15}$$

From this there result the proper frequencies

$$\omega^2_{ms} = g k_{ms} \tanh k_{ms} h \{1 + \sigma k^2_{ms}/(\rho g)\}. \tag{16}$$

The contribution of surface tension is unimportant for the lower modes of vibration ($k_{ms} R$ small) except in cylinders of quite small radius. Since it adds only 1% to the frequency of the mode considered in the smallest cylinder used in the measurements ($R = 1{\cdot}5$ in.), it will henceforth be omitted, and Ω will be approximated by Ω_g (given by (13 a)).

In the above development there is, of course, no damping. To find this we turn to the linearized Navier–Stokes equation for an incompressible fluid

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla\left(gz + \frac{p}{\rho}\right) + \nu \nabla^2 \mathbf{u}, \tag{17}$$

with the equation of continuity

$$\nabla . \mathbf{u} = 0, \tag{18}$$

and with the boundary condition

$$\mathbf{u} = 0 \tag{19}$$

on fixed boundaries. Here p is the pressure and $\nu = \mu/\rho$ the kinematic viscosity.

Let us represent the velocity vector in the form

$$\mathbf{u} = -\nabla\phi + \nabla \times \mathbf{A}, \tag{20}$$

where ϕ and $\mathbf{A}$ are respectively scalar and vector functions of (r, θ, z, t). It is readily demonstrated that this, together with

$$\frac{p}{\rho} + gz = \frac{\partial\phi}{\partial t} + \text{const.}, \tag{21}$$

constitutes a solution of (17) and (18), provided that

$$\nabla^2\phi = 0, \tag{22}$$

$$\nu\nabla^2\mathbf{A} = \frac{\partial \mathbf{A}}{\partial t}. \tag{23}$$

If the boundary conditions of the problem can be satisfied by means of the expression (29) subject to (22) and (23), we may infer that this is the complete solution. The boundary conditions at the rigid surfaces S

$$(-\nabla\phi + \nabla \times \mathbf{A})_S = 0, \tag{24}$$

together with (22), (23) and the surface condition (10) make possible, in principle, an explicit, rigorous solution for ϕ and $\mathbf{A}$ in terms of the surface coordinates. However, the general experience gained with boundary layers in the last fifty years suggests that this is hardly necessary. It is to be expected that ϕ is essentially the velocity potential ϕ_c characteristic of inviscid flow (i.e. the potential for which $(\partial\phi_c/\partial n)_S = 0$).

Let us write $\phi = \phi_c + \phi_c'$. We shall see that the contribution of the additional part ϕ_c' to the velocity is quite small. Assuming this (subject to later verification), we have as approximate boundary conditions on the rigid surfaces

$$[\mathbf{n} \times (\nabla \times \mathbf{A})]_S = (\mathbf{n} \times \nabla\phi_c)_S, \tag{25 a}$$

and

$$(\mathbf{n} \,.\, \nabla \times \mathbf{A})_S = (\partial\phi_c'/\partial n)_S. \tag{25 b}$$

Since ϕ_c is now assumed known, (23) and (25 a) determine $\mathbf{A}$. Having obtained $\mathbf{A}$, (22) and (25 b) determine ϕ_c'. The method of calculation is, in principle at least, one of successive approximation. Starting with the zeroth approximation ($\mathbf{A} = 0$, $\phi = \phi_c$), we calculate a first approximation to $\mathbf{A}$. With this a correction to ϕ can be found. In principle this process could be repeated to an arbitrary accuracy. In practice, however, the first approximation is sufficiently accurate by itself.

The solution of the equations for $\mathbf{A}$ is enormously simplified by noting that the region of non-vanishing $\mathbf{A}$ is confined to the immediate vicinity of the boundaries. Indeed, assuming the special case of a simple harmonic time dependence ($\mathbf{A} \propto e^{-i\omega t}$), we obtain

$$(\nabla^2 + i/l^2)\mathbf{A} = 0, \tag{26}$$

where the boundary-layer thickness $l = \sqrt{(\nu/\omega)}$ is of the order of 0·1 mm for the values of ν and ω arising in the experiments described later. Outside this layer, $\mathbf{A}$ vanishes exponentially. Consequently, to find $\mathbf{A}$ near a given boundary, the other boundaries can be ignored. For a standing wave this case represents a forced oscillation, requiring a supply of energy to counteract viscous dissipation. However, for free oscillations in which the damping is small, the frequency ω is large in comparison with α in the exponential decay factor $e^{-\alpha t}$; thus we may again assume a boundary layer of thickness $\sqrt{(\nu/\omega)}$, and may make use of the suggested approximations. In the body of the liquid, including the vicinity of the free surface, $\mathbf{A}$ can be taken to be zero in this approximation; accordingly, the boundary conditions at the free surface are satisfied in terms of ϕ_c alone.

To compute the viscous dissipation we use the well-known result (see Lamb (1945), § 329)

$$d\overline{E}/dt = -2\overline{F} \tag{27}$$

where E is the total energy $(T+\Omega)$, and $2F$ is the dissipation function given by

$$2F = \mu \int_V (\nabla \times \mathbf{u})^2 \, dV + \mu \int_{S'} (\mathbf{n} \,.\, \nabla \mathbf{u}^2) \, dS - 2\mu \int_{S'} \mathbf{n} \,.\, \mathbf{u} \times (\nabla \times \mathbf{u}) \, dS, \tag{28}$$

in which the two surface integrals extend over the whole boundary $S' = S + S_t$ of the liquid volume V. The bars denote averages over a cycle for an assumed harmonic time dependence. Equation (28) simplifies on noting that $\mathbf{u} = 0$ on rigid boundaries, and $\nabla \times \mathbf{u} = 0$ approximately at the free surface. Therefore

$$2F = 2F_r + 2F_t \tag{29}$$

approximately, where

$$2F_r = \mu \int_V (\nabla \times \nabla \times \mathbf{A})^2 \, dV, \tag{30}$$

and

$$2F_t = -\mu \int_{S_t} \frac{\partial}{\partial n} (\nabla \phi_c)^2 \, dS, \tag{31}$$

where $\mathbf{A}$ is expected to be insignificant away from the solid boundaries. The latter integral, which is a simple transformation of a volume integral, represents the dissipation in the body of the liquid. It may at first seem inconsistent to retain this term, which is clearly of smaller order of magnitude than the first term under the assumptions of the present approximation, while neglecting the dissipation, due to rotational motion at the free surface, represented by the third integral in (28). (This integral measures the additional dissipation in the 'boundary layer' which must exist near the free surface in order that viscous stresses are balanced at this surface—where the only possible non-uniform stress is that of surface tension.) However, the dissipation in the boundary layer at the free surface is in fact of lesser order of magnitude than that in the body of the liquid (see Ursell (1952), p. 94), and the term (31) gives a reliable indication of the magnitude of the dissipation occurring away from the solid boundaries.

To find the damping of a single mode, we keep only one term in (8). The kinetic and potential energies are then

$$T = \tfrac{1}{2}\rho \dot{q}_{ms}^2 \frac{\coth k_{ms} h}{k_{ms}}, \tag{12 b}$$

$$\Omega = \tfrac{1}{2}\rho g q_{ms}^2. \tag{13 b}$$

For damped oscillations we have

$$q_{ms} = q_{ms}^* e^{-i\omega_{ms}t-\alpha t}. \tag{32}$$

Assuming $\alpha \ll \omega$ (which is justifiable *a posteriori*) we obtain

$$\frac{d\bar{E}}{dt} = \rho g q_{ms}^{*2} (-\alpha) e^{-2\alpha t}. \tag{33}$$

Inserting ϕ_c from (8) into (31) we obtain after some elementary integrations

$$2F_l = 2\mu k_{ms}^2 \rho g q_{ms}^{*2} e^{-2\alpha t}. \tag{34}$$

The differential equation for **A** in the vicinity of a given wall can be integrated readily in terms of known functions (assuming the other boundaries to be at infinity). However, even this is not necessary. The boundary layer thickness l is small compared to the radius of the cylinders considered. Hence we can neglect the effects of curvature. (The error made is then of the order of l/R, which is less than 1%.) With this approximation the vector potential in the vicinity of the side walls is

$$\left.\begin{aligned} A_r &= D_r e^{-\sqrt{i}(R-r)/l} \cos s\theta \sinh k_{ms}(z+\tfrac{1}{2}h), \\ A_\theta &= D_\theta e^{-\sqrt{i}(R-r)/l} \sin s\theta \sinh k_{ms}(z+\tfrac{1}{2}h), \\ A_z &= 0, \end{aligned}\right\} \tag{35}$$

where

$$\left.\begin{aligned} D_r &= \frac{s}{k_{ms}R} J_s(k_{ms}R)\left(\frac{-\dot{q}_{ms}N_{ms}}{k_{ms}\sinh k_{ms}h}\right), \\ D_\theta &= \frac{l}{\sqrt{i}} k_{ms} J_s(k_{ms}R)\left(\frac{-\dot{q}_{ms}N_{ms}}{k_{ms}\sinh k_{ms}h}\right)\left(1-\frac{s^2}{(k_{ms}R)^2}\right), \\ l &= \sqrt{(\nu/\omega_{ms})}. \end{aligned}\right\} \tag{36}$$

With this expression for **A** it can be seen from (25 b) that the resulting ϕ_c' is $O(lk_{ms}) = O(l/R)$. This then verifies the earlier statement that the contribution of ϕ_c' to the fluid velocities is small. Indeed, we see that the approximation method adopted is essentially an expansion in powers of l/R.

Correct to first order in l/R, we find from (35) in conjunction with (30) the contribution $(2F_s)$ of the side walls to the dissipation due to the rigid boundaries to be

$$2\bar{F}_s = \frac{\mu\omega_{ms}^2}{k_{ms}R} \frac{q_{ms}^{*2} e^{-2\alpha t}}{[1-(s/k_{ms}R)^2]} \frac{1}{2\sqrt{2l}} \times$$
$$\times \left\{(\coth k_{ms}h)[1+(s/k_{ms}R)^2] - \left(\frac{k_{ms}h}{\sinh^2 k_{ms}h}\right)[1-(s/k_{ms}R)^2]\right\}. \tag{37}$$

The contributions to the dissipation from the tank bottom are obtained similarly. Thus, for the vector potential in the vicinity of $z = -\frac{1}{2}h$ we find

$$\left.\begin{aligned} A_r &= Ee^{-\sqrt{i}(z+\frac{1}{2}h)/l}\left(\frac{s}{k_{ms}R}\right)J_s(k_{ms}r)\cos s\theta, \\ A_\theta &= -Ee^{-\sqrt{i}(z+\frac{1}{2}h)/l}\left(\frac{1}{k_{ms}}\right)\frac{d}{dr}J_s(k_{ms}r)\sin s\theta, \\ A_z &= 0, \end{aligned}\right\} \tag{38}$$

where

$$E = \frac{-l}{\sqrt{i}}\left(\frac{-\dot{q}_{ms}N_{ms}}{\sinh k_{ms}h}\right). \tag{39}$$

Calling the contribution to the dissipation $2F_b$ we have (again to first order in l/R)

$$2\bar{F}_b = \frac{\mu k_{ms}\overset{*}{q}{}_{ms}^{2}\,ge^{-2\alpha t}}{\sqrt{2}l\sinh 2k_{ms}h}. \tag{40}$$

Inserting the results expressed in (33), (34), (37), and (40) into (27) yields an equation for the damping constant α. Clearly this is a sum of three terms corresponding to the damping of the free surface (34), the side walls (37) and the bottom (40).

A conventional description of the damping is in terms of the logarithmic decrement $\delta = 2\pi\alpha/\omega$. The result is

$$\delta = \delta_t + \delta_s + \delta_b, \tag{41}$$

where

$$\delta_t = \frac{4\pi\nu k_{ms}^2}{\omega_{ms}}, \tag{42 a}$$

$$\delta_s = \left(\frac{\nu}{2\omega_{ms}}\right)^{\frac{1}{2}}\left(\frac{\pi}{R}\right)\left\{\frac{1+(s/k_{ms}R)^2}{1-(s/k_{ms}R)^2} - \frac{2k_{ms}h}{\sinh 2k_{ms}h}\right\}, \tag{42 b}$$

$$\delta_b = \left(\frac{\nu}{2\omega_{ms}}\right)^{\frac{1}{2}}\left(\frac{\pi}{R}\right)\frac{2k_{ms}R}{\sinh 2k_{ms}h}. \tag{42 c}$$

The case which we have examined in detail is the fundamental mode with $s = 1$, $k_{ms} = 1{\cdot}841R^{-1}$. In figure 2, δ is plotted as a function of h/R for several R; numerical results are given in table 1.

Three remarks are perhaps in order.

(*a*) As would be expected since most of the dissipation takes place at the rigid boundaries, the damping is rather insensitive to the shape of the cylinder cross-section except in so far as the total area of the walls is affected. Thus, the δ for the circular cylinder is only 18% less than that for a square cylinder of the same cross-sectional area.

(*b*) The damping in the body of the liquid (42 a) is small compared to the wall damping—particularly for large cylinders. Indeed, in the sense of an expansion in powers of l/R one should perhaps not retain this term. It is readily seen that δ_t is $O(l^2/R^2)$ while δ_s and δ_b are $O(l/R)$. However, there are two reasons for keeping it. First, it permits an evaluation of the relative contributions to the damping. Thus, for the smallest of the cylinders used, the contribution of this term to the damping was roughly 10%. Second, in the limit of an infinitely deep ocean infinite in extent this is the only damping term (cf. Lamb (1945), § 348). (Of course, then k_{ms} stands for the wave number of whatever wave is being considered and is not necessarily zero.)

(*c*) The method of calculation adopted is probably not the simplest in order to obtain results to the given order in l/R. It was used since it is a logical first step in a seemingly convergent sequence of successive approximations, and permits a clear estimation of the errors remaining after the first approximation.

The comparison of the theoretical prediction (42) with the experiments is given below.

III. Experimental tests

Measurements were made of the logarithmic decrement as a function of h/R for tap water at 16° C in brass cylinders of radius 1·5 in. and 3 in., and a steel cylinder of radius 10 in. The brass cylinders were made from 1/8 in. wall extruded brass tubing, the steel cylinder from 3/32 in. sheet rolled to a 10 in. radius and brazed. Bases of 3/8 in. thick brass, fitted to each of the cylinders by grooving and silver-soldering, were of square cross section and cut tangent to the cylinder.

The amplitude of the surface oscillation was recorded as a function of time by means of a sensing element in the liquid, connected through a low-frequency amplifier to an Esterline–Angus recorder. The sensing element consisted of a 40-gauge wire stretched taut parallel to the axis of the cylinder 1/16 in. from the wall and insulated from the cylinder by a bushing in the base. The wire was located accurately on that diameter of the cylinder which was perpendicular to one edge of the base. To obtain good linearity and stability of response, the wire was connected in series with the cathode of the first stage of the amplifier, the conduction path for the vacuum tube being completed through the liquid to the grounded cylinder. Since the resistance between the wire and the cylinder is proportional (very nearly) to the length of wire below the free surface, the change in voltage between cathode and grid of the first amplifier stage is proportional to the displacement of the free surface of the fluid from equilibrium. To obtain greater sensitivity for the larger values of h/R, the wire was extended only 2 in. below the free surface of the liquid, a rubber band between the wire and the base serving to keep the wire taut and to insulate it. Two such wires, each with an amplifier and recorder, were used in each tube and were placed accurately 90° apart. One recorded the motion of the fundamental transverse mode, while the other, being placed on the nodal diameter, served as a monitor to indicate the purity of the transverse modes. The mechanical arrangement is illustrated in figure 1.

The deflection of the recorder was calibrated in terms of the amplitude of oscillation of the fluid by means of a sliding probe fitted with a needle at the lower end and carrying a scale engraved in 0·05 in. divisions. The probe, located diametrically opposite the sensing element and parallel to, and supported from, the wall of the cylinder, was connected electrically in series with a generator of 1000 cycle/sec frequency and headphones, the final adjustment of the water level being made with an eyedropper. The probe was then raised a given distance and the fundamental mode excited by rocking the cylinder about the edge of the base. The gain of the amplifier was adjusted until the deflection of the recorder equalled a convenient number of scale divisions at the time the last intermittent tone was heard as the oscillations damped out. By setting the probe at various distances above the equilibrium surface, the response of the recording system was determined and found to be linear within the accuracy of the measurements.

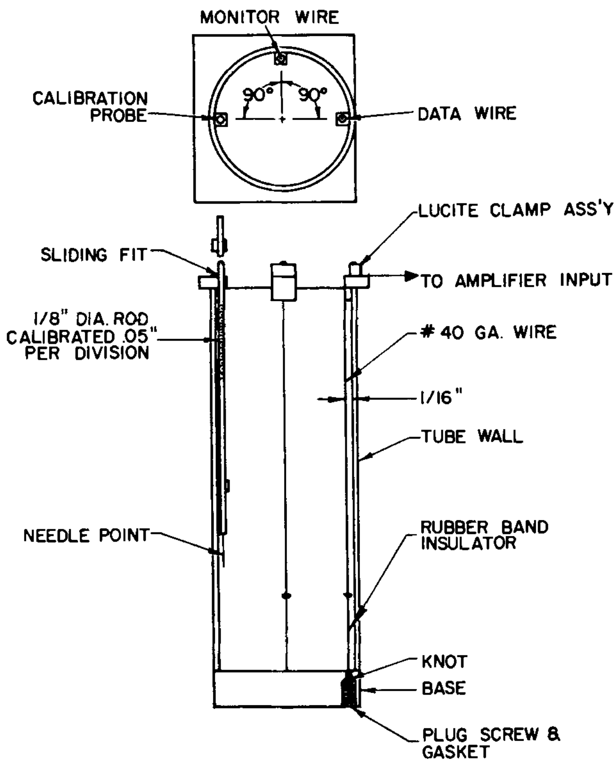

Figure 1. Arrangement of the sensing elements and calibration probe. The walls of the brass tube are $\frac{1}{8}$ in. thick, and the base is of 3 in. brass plate.

Measurements of the logarithmic decrement were made for a range of amplitudes, the largest corresponding to a/R approximately equal to 0·2. It was difficult to excite larger amplitudes without exciting higher modes. However, up to this limit, no deviations from a simple exponential decay were observed.

Figure 2 and table 1 give the results of the measurements for the 1·5 in. and 3 in. radius tubes. For the number of trials made at each value of

h/R the standard deviation of the mean was less than 3% except for $h/R = 0{\cdot}25$ for the 1·5 in. tube; for this it was 9%. The results for the 10 in. radius tube are not included for reasons given below. The build-up of a circular mode was essentially eliminated by stressing the cylinder slightly to cause the major axis of the ellipse to coincide with the diameter containing the sensing element. The eccentricities required to maintain the transverse mode were of the order of a few thousandths of an inch.

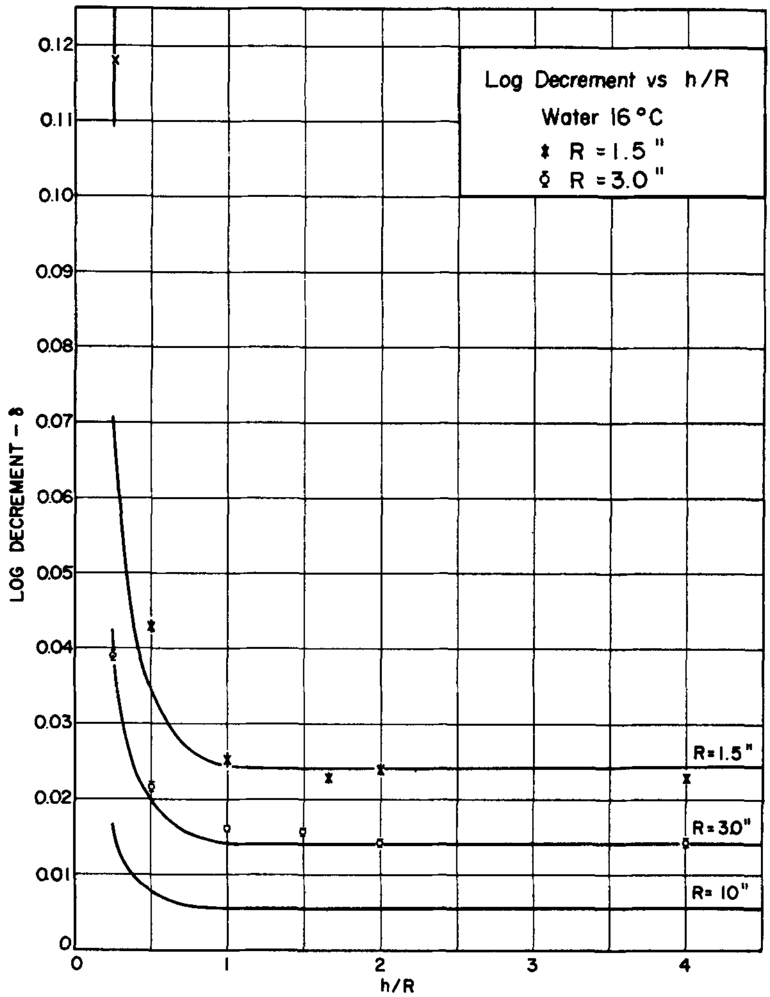

Figure 2. Comparison of the theoretical and experimental results for the damping of surface oscillations in polished right circular cylinders.

Initial measurements of the decrement with the surfaces unpolished gave values of the logarithmic decrement too large by a factor of between 2 and 3, as shown in figure 3. After polishing the tubes to a mirror finish by hand in a direction parallel to the axis of the cylinder the results shown in figure 2 were obtained.

The bottom surface of the 1·5 in. radius cylinder could not be reached by hand and was polished in a lathe. This process leaves minute circular grooves, which are believed to account for the large values of δ_0 observed for $h/R = 0{\cdot}25$ and $0{\cdot}5$, since it is for these values of h/R that the contribution of the bottom to the damping is greatest. The decrement for the fundamental mode in this region is actually larger than for the second mode, making it more difficult to obtain a pure fundamental. Contamination of the inside surface with wax, oil, or even the film left by evaporating alcohol increases the decrement by several per cent.

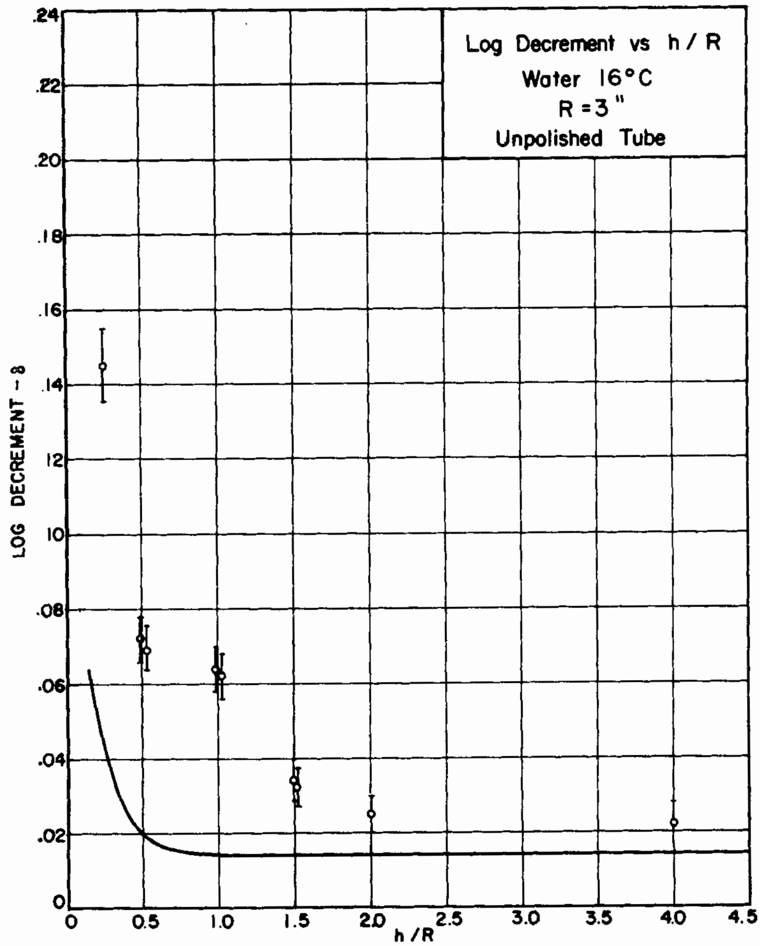

Figure 3. Comparison of the theory (solid curve) with measurements in an unpolished cylinder.

The 10 in. radius tube, because of the nature of the steel, could not be polished to the required degree. The measured decrements were again a factor of between 2 and 3 too large, and for this reason the results are not tabulated. The shape of the curve obtained was similar to those of figure 2.

IV. Comparison of theory and experiment

The overall agreement between theory and experiment (figure 2 and table 1) is rather satisfactory and would seem to show that the analysis given above is adequate.

R (in.)	h/R	f_c (sec^{-1})	f_0	f_c/f_0	$\delta_c \times 100$	δ_0	δ_{cc}	δ_c/δ_0	δ_{cc}/δ_0
1·5	0·25	2·27	2·26	1·005	7·06	11·8	7·08	0·598	0·600
,,	0·50	2·95	3·04	0·970	3·45	4·27	3·40	0·808	0·796
,,	1·0	3·37	3·62	0·931	2·47	2·52	2·385	0·891	0·947
,,	1·667	3·46	3·76	0·921	2·42	2·28	2·32	1·06	1·02
,,	2·0	3·46	3·77	0·919	2·42	2·38	2·32	1·02	0·976
,,	4·0	3·46	3·80	0·911	2·44	2·27	2·33	1·07	1·025
3	0·25	1·61	1·73	0·932	4·15	3·89	4·00	1·065	1·025
,,	0·5	2·08	2·16	0·964	2·02	2·17	1·98	0·933	0·914
,,	1·0	2·39	2·47	0·968	1·44	1·61	1·415	0·895	0·880
,,	1·5	2·44	2·52	0·968	1·40	1·55	1·38	0·904	0·891
,,	2·0	2·45	2·55	0·962	1·41	1·41	1·38	1·02	0·980
,,	4·0	2·45	2·56	0·959	1·42	1·41	1·38	1·03	0·980

Table 1. Comparison of theory and experiment. Surface oscillations in right circular cylinders.

One discrepancy should perhaps be noted. From table 1 one sees that the observed frequencies are somewhat higher than those calculated from the classical formulae. We believe this can be understood on the basis of surface tension effects associated with wetting of the wall. (This does not mean the surface tension effects associated with the main part of the free surface. This we have seen is much too small.) A crude analysis suggests that the correct logarithmic decrement should again be given by (42) using as the frequency ω the observed frequency. A check of this is obtained on noting that one would expect the observed decrement always to be greater than the calculated value. (All dirt effects would presumably tend to increase the dissipation and hence to increase the observed decrement.) In table 1 the ratio of theoretical decrement corrected in this way (δ_{cc}) to the observed (δ_0) is always (within the standard deviation of the measurements) less than, or equal to, unity.

The extreme sensitivity of the experimental results to the condition of the walls is quite interesting. Roughnesses whose depth was small compared to the boundary layer thickness had a remarkably large effect.

V. Conclusion

It would seem that on the basis of these results one can conclude that an essentially correct description of the damping is given by equation (42). This is to be understood in the sense of rather ideal situations with very smooth walls. Depending on the roughness of the walls, there is an additional factor of between 2 and 4 in the decrement which is approximately size independent. This serves to emphasize the extreme care which must be used in comparing numerical results of experiments of this kind with theoretical predictions.

Two questions raised by this work are left unanswered.

(*a*) Can one understand in detail the effects of very small roughness on the damping?

(*b*) Can one improve the frequency calculation to obtain closer numerical agreement with the observed frequencies?

It is a pleasure to acknowledge the considerable assistance given by Mr Ronald Green during the carrying out of the experimental work. One of us (K. M. C.) would like to thank the Institute for Advanced Study at Princeton for its hospitality and a grant-in-aid during the period when this report was written. Work related to this problem has been done by the authors under the auspices of the Ramo-Wooldridge Corporation, Los Angeles, California.

References

Biesel, F. 1949 *La Houille Blanche* **4**, 630.

Boussinesq, J. 1878 *J. Math. Pures Appl.* **4**, 335.

Hunt, J. N. 1952 *La Houille Blanche* **7**, 836.

Lamb, H. 1945 *Hydrodynamics*, 6th Ed. New York : Dover.

Keulegan, G. H. 1948 *J. Res. Nat. Bur. Stand., Wash.*, **40**, 487.

Ursell, F. 1952 *Proc. Roy. Soc.* A, **214**, 79.